

Vassiliev knot invariants and Chern-Simons perturbation theory to all orders

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Abstract

At any order, the perturbative expansion of the expectation values of Wilson lines in Chern-Simons theory gives certain integral expressions. We show that they all lead to knot invariants. Moreover these are finite type invariants whose order coincides with the order in the perturbative expansion. Together they combine to give a universal Vassiliev invariant.

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1 Introduction

Chern-Simons theory is the most popular example of topological field theory in 3 dimensions. Given a compact Lie group G , a compact, oriented 3-manifold M , a link $L \subset M$, and for each component of L a representation of G , this theory associates topological invariants to these data. There are several ways to define the invariants, which are all closely related. First of all there are the non-perturbative definitions: Witten [18] used fundamental properties of quantum field theory, in particular the path integral formulation, and Reshetikhin and Turaev [15] used quantum groups. These two definitions are equivalent.

Then there are the perturbative definitions, the first of which were given by Guadagnini et al. [10] in the case $M = S^3$, $L \neq \emptyset$, using propagators and Feynman diagrams. This approach was then elaborated by Bar-Natan [3, 4]. The case $M \neq S^3$, $L = \emptyset$ was treated by Axelrod and Singer [2]. A common feature of all these works is the Feynman diagram expansion familiar in perturbative quantum field theory. Invariants are defined at every order in the expansion, each is a sum of several terms corresponding to the diagrams of the given order. The contribution of any diagram is the product of two factors, the first depends only on the group G and the representations associated to the components of L , and the second is independent of G and its representations, it is an integral over the configuration space of the vertices of the diagram, some of which are constrained to lie on L , while the others can lie anywhere in the complement of L . When L is a knot in S^3 , several properties of the invariant arising from the contributions of order two were already discussed in [10], although the invariance itself was shown in [4]. Bar-Natan also studied the properties of the group-dependent contributions, and among them he found relations between the contributions of different diagrams which are the same for all groups G . This led him [5] to define abstract objects, which we call BN diagrams, by these relations, and abstract invariants which take their values in the space of BN diagrams. To every choice of group G and representations corresponds a linear functional on the space of BN diagrams. Applying this functional to the abstract invariants gives back the ordinary group-dependent invariants.

In order to show that the contributions of a given order sum up to an invariant, one must compute the variation of these integrals under a small change of the embedding of L , and this proved to be quite difficult and lengthy. However, Bott and Taubes [7] greatly improved this situation. They showed that the variation can be split in two terms, the “diagrammatic” and the “anomalous” variations. As its name indicates, the diagrammatic variation can be read at once from the Feynman diagram. It corresponds to the differential of Kontsevich’s graph complex, obtained by collapsing the edges. The anomalous variation is more difficult to compute, but it is proportional to the variation of the first order contribution,

the “self-linking number”. The constant of proportionality, is still unknown in general, but independent of the embedding. These results of Bott and Taubes are powerful enough, as we will show, to prove invariance at all orders.

During the same period, the subject of Vassiliev knot invariants, also known as finite type invariants, was developing rapidly. The starting point of Vassiliev [16] was the space of all immersions of S^1 in S^3 . In this space, a knot type is a cell whose faces are singular knots with a finite number of transverse double points. Any knot invariant can be extended to such singular knots. It is said to be a finite type invariant of order $\leq N$, if it vanishes on all singular knots with more than N double points. Let V^N be the space of invariants of order $\leq N$. Unexpectedly at first, Bar-Natan found that V^N/V^{N-1} embeds in the dual of the space of BN diagrams of degree N . Kontsevich [13] showed that the two spaces are in fact isomorphic. His proof [5] involved the construction of a universal Vassiliev invariant, a formal power series in the space of BN diagrams whose coefficients are finite type invariants, based on the Knizhnik-Zamolodchikov equations of the WZW model of conformal field theory. (It was Witten [18] who discovered the relation between conformal field theory and topological field theory. Fröhlich and King [9] were the first to construct link invariants from the KZ equations.)

In this paper we start from the results of Bott and Taubes [7] to construct a universal Vassiliev knot invariant, given by the perturbative expansion of the expectation value of a Wilson loop in Chern-Simons theory on \mathbb{R}^3 . The basic ingredient in the integrals obtained from the Feynman rules is the propagator of the gauge field, which is given in the Lorentz gauge by the Gauss two-form, the pullback of the volume form on S^2 . During the Aarhus conference on geometry and physics in the summer of 1995, we learned from Dylan Thurston that he independently obtained similar results to ours [17]. Recently, perturbative Chern-Simons invariants have also been investigated in [1, 11].

In more details, the contents of the paper are as follows: in section 2, we define the graphs appearing in the perturbative expansion, which are equipped with an additional structure called vertex orientation, and state some simple combinatorial lemmas. In section 3 we give the Feynman rules, in which the vertex orientation plays an important role. They allow us to define unambiguously the signs of the contributions of graphs appearing in the perturbative expansion (see propositions 2 and 3). In section 4 we define the expectation value of a Wilson loop Z , which is a sum over trivalent graphs, and prove that it is invariant under the changes of embedding corresponding to the collapse of a single edge (theorem 1). In section 5 we consider the other variations of the embedding, called “anomalous”. We improve some results of [7] in proposition 3 and theorem 2, which allow us to conclude in theorem 3 that a suitably corrected version of Z becomes a framed knot invariant \hat{Z} . In the last section

we prove that \hat{Z} is a universal invariant. In particular, the N -th order contribution to \hat{Z} is a finite type invariant of order $\leq N$. Although it is stated explicitly in the literature, we have never seen a proof of this going beyond the second order. The question whether the KZ and the Chern-Simons universal invariants are equal is still open. The answer would be positive if one could show that the Chern-Simons invariant extends functorially to the category of tangles, as in the case of the KZ invariant, but at least to us it is not obvious that it has this extension property. In an appendix, we recall the definition of the pushforward, or integration along the fiber, which enters the formulation of the Feynman rules.

2 Graphs

A Wilson graph Γ is a one-dimensional, connected, simplicial complex equipped with some additional structures, which we now describe. We assume that all vertices have valence ≥ 3 and that each graph has a distinguished oriented cycle W_Γ , called the Wilson line. We define:

$$\begin{aligned} V_\Gamma &= \{\text{vertices of } \Gamma\}, & E_\Gamma &= \{\text{edges of } \Gamma\}, \\ V_\Gamma^o &= V_\Gamma \cap W_\Gamma, & E_\Gamma^o &= \{e \in E_\Gamma \mid e \not\subset W_\Gamma, \partial e = e \cap W_\Gamma\}, \\ V_\Gamma^i &= V_\Gamma - V_\Gamma^o, & E_\Gamma^a &= E_\Gamma - E_\Gamma^o, \\ E_\Gamma^W &= E_\Gamma \cap W_\Gamma, & E_\Gamma^{nW} &= E_\Gamma - E_\Gamma^W. \end{aligned} \tag{2.1}$$

E_Γ^a is called the set of admissible edges of Γ , V_Γ^o the set of external vertices, V_Γ^i the set of internal vertices. An example of Wilson graph is given in fig. 1. The edges of E_Γ^W are solid lines, those of E_Γ^{nW} are dashed lines. We say that two Wilson graphs are equivalent, and

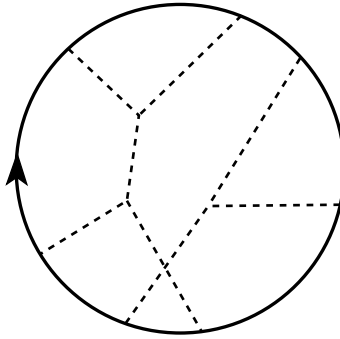


Figure 1:

write $\Gamma \sim \Gamma'$, if there are two bijections

$$\rho : V_\Gamma \longrightarrow V_{\Gamma'}, \quad \sigma : E_\Gamma \longrightarrow E_{\Gamma'}, \tag{2.2}$$

such that $\sigma|_{E_\Gamma^W}$ is a map from E_Γ^W to $E_{\Gamma'}^W$ preserving the orientation of the Wilson line, and

$$\rho \partial_\Gamma = \partial_{\Gamma'} \sigma, \quad (2.3)$$

where ∂_Γ is the usual boundary map acting on E_Γ . The pair (ρ, σ) is an isomorphism of graphs. It is an automorphism if $\Gamma = \Gamma'$. Let $\text{Aut}(\Gamma)$ be the finite group of automorphisms of Γ .

We define an additional structure on graphs which we call vertex orientation, or V-orientation, which should not be confused with the usual definition of graph orientation. We orient Γ by giving for each vertex an order among the edges arriving at this vertex. More precisely for each vertex $v \in V_\Gamma$ we choose a bijection :

$$o_v : E_v \longrightarrow \{1, 2 \cdots, |E_v|\}, \quad (2.4)$$

where $E_v = \{e \in E_\Gamma^W \mid v \in \partial e\}$. We say that $O_\Gamma = \{(o_v)_{v \in V_\Gamma}\}$ and $O'_\Gamma = \{(o'_v)_{v \in V_\Gamma}\}$ define the same (resp. opposite) V-orientation if

$$\prod_{v \in V_\Gamma} \text{sign}((o_v)^{-1} o'_v) = +1 \text{ (resp. } -1), \quad (2.5)$$

where sign is the signature of a permutation. For short, in the sequel Γ will denote the graph Γ with a choice of V-orientation O_Γ , and $-\Gamma$ the graph equipped with the opposite V-orientation.

Using our definition of isomorphism of graphs, we can define the notion of induced orientation: $(\rho, \sigma)O_\Gamma = \{(o_v \circ \sigma)_{v \in V_\Gamma}\}$. Since Γ is determined by the couple $(\partial_\Gamma, O_\Gamma)$ we put

$$(\rho, \sigma) \cdot \Gamma = (\rho \partial_\Gamma \sigma^{-1}, (\rho, \sigma)O_\Gamma). \quad (2.6)$$

The equivalence relation naturally extends to these couples.

Thus we can say that an automorphism preserves the orientation if the induced orientation coincides with the original one, and denote by $\text{Aut}_+(\Gamma)$ the normal subgroup of orientation preserving automorphisms. In the sequel we will deal only with trivalent and tetravalent vertices. In this case it is possible to give a graphical representation of V-orientation. For trivalent vertices the orientation is given by a cyclic order, see fig. 2. For internal tetravalent vertices, because the signature of a cyclic permutation of 4 objects is -1 , the orientation is given by a separation between the first and the last edges, see fig. 3. For Wilson tetravalent vertices, the orientation is given by an order between the internal edges, see fig. 4. We will not draw the cyclic orientation of trivalent vertices if they all agree with the orientation of the Wilson loop.

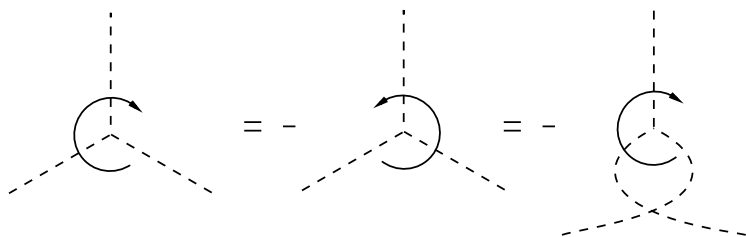


Figure 2:

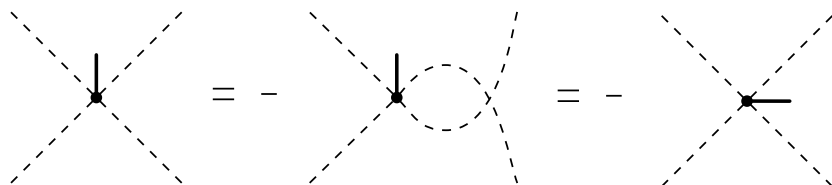


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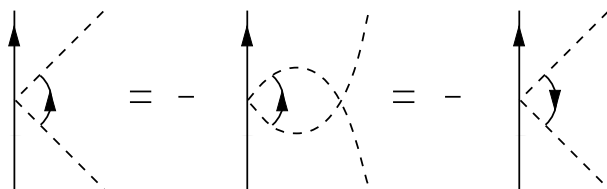


Figure 4:

Let Γ be a trivalent graph, and $e \in E_\Gamma^a$ an admissible edge. We define $\delta_e \Gamma$ to be the graph obtained from Γ by collapsing e to a 4-valent vertex x , and the vertex orientation of $\delta_e \Gamma$ is defined by the following rule:

If $e \cap W_\Gamma = \emptyset$ we define it by fig. 5. If $e \cap W_\Gamma \neq \emptyset$ and $e \not\subset W_\Gamma$ we define it by fig. 6.

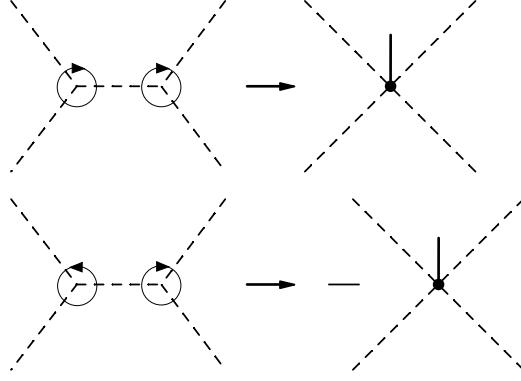


Figure 5:

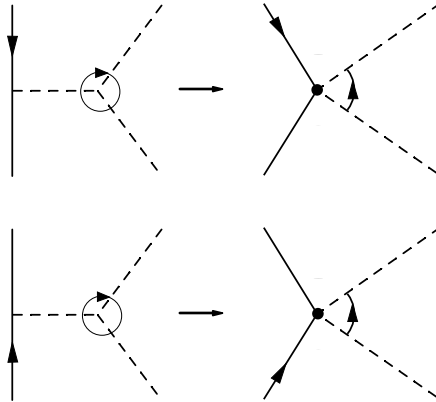


Figure 6:

If $e \subset W_\Gamma$ we define it by fig. 7, where the oriented edges are those of W_Γ .

Lemma 1 *Let Γ_x be a graph with one 4-valent vertex x , and all other vertices trivalent. Then there are at most three 3-valent graphs Γ_x^i , $i = 1, 2, 3$, and edges $e_i \in E_{\Gamma_x^i}$ such that $\delta_{e_i} \Gamma_x^i = \Gamma_x$. If the vertex x of Γ_x looks like fig. 8, then the vicinity of the edges e_i in Γ_x^i is given by fig. 9.*

However, it may happen that $\Gamma_x^i \sim \Gamma_x^j$ for $i \neq j$.

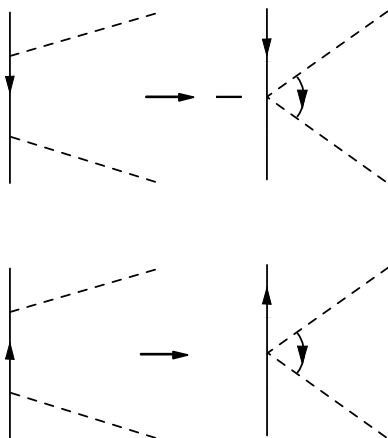


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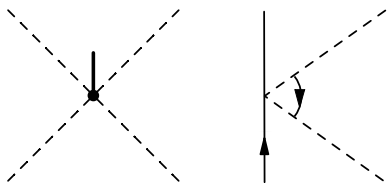


Figure 8:

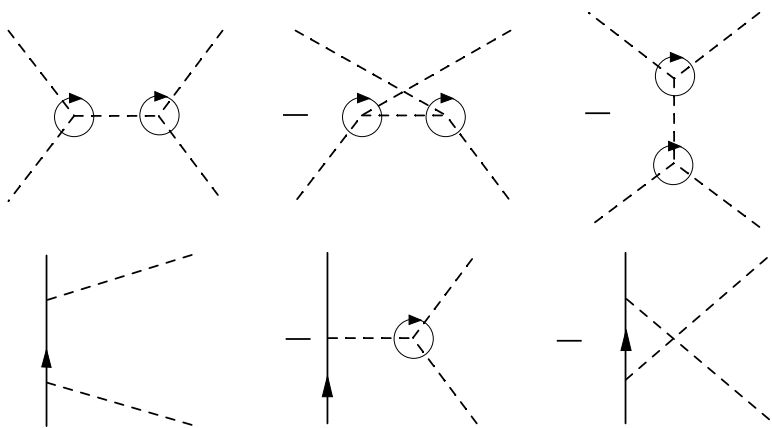


Figure 9:

Lemma 2 *Let Γ a trivalent graph $e, f \in E_\Gamma^a$. Then $\delta_e \Gamma \sim \delta_f \Gamma$ iff there exists $(\rho, \sigma) \in \text{Aut}(V_\Gamma) \times \text{Aut}(E_\Gamma)$ such that*

$$\begin{aligned}\delta_e((\rho, \sigma) \cdot \Gamma) &= \delta_e \Gamma, \\ \sigma(f) &= e.\end{aligned}\tag{2.7}$$

We omit the proofs of these lemmas, which are essentially obvious. Later we shall use the following combination of the two lemmas.

Proposition 1 *Let Γ be a trivalent graph, $e, f \in E_\Gamma^a$. Then $\delta_e \Gamma \sim \pm \delta_f \Gamma$ iff there exists $i \in \{1, 2, 3\}$ and $(\rho, \sigma) \in \text{Aut}(V_\Gamma) \times \text{Aut}(E_\Gamma)$ such that*

$$(\rho, \sigma) \cdot \Gamma = \pm \Gamma_x^i, \quad \text{and } \sigma(f) = e, \text{ where } \Gamma_x = \delta_e \Gamma.\tag{2.8}$$

It is easy to see that if Γ is a trivalent graph with n external vertices and t internal vertices, $n + t$ is even. The degree (order) of Γ is defined to be:

$$\deg \Gamma = (n + t)/2.\tag{2.9}$$

Let G^3 be the set of equivalence classes of vertex-oriented trivalent graphs. Consider the vector space $\tilde{\mathcal{A}}$ over \mathbb{R} with basis elements $\Gamma \in G^3$, and \mathcal{A} the quotient of $\tilde{\mathcal{A}}$ by the subspace spanned by all the vectors

$$\begin{aligned} (*) \quad & \sum_{i=1}^3 \Gamma_x^i \\ (**) \quad & (\Gamma) + (-\Gamma),\end{aligned}\tag{2.10}$$

where Γ_x is any graph with a single tetravalent vertex. We call \mathcal{A} the space of BN diagrams. Denote by D the canonical projection from $\tilde{\mathcal{A}}$ to \mathcal{A} . Note that $(*)$ coincides with the IHX and STU relations of Bar-Natan [3, 5].

It was shown in [5] that \mathcal{A} is a commutative associative algebra, with the product obtained by taking the connected sum along the Wilson lines of two graphs. A *prime* Wilson graph is a graph which cannot be expressed as a product of two non-trivial graphs. Equivalently, a prime graph is such that if we cut it along any two edges of the Wilson line, the resulting graph is connected. A graph Γ is *primitive* if $\Gamma - W_\Gamma$ is connected. Of course, a primitive graph is prime (see examples in fig. 10). Later we will need a lemma due to Bar-Natan [3]:

Lemma 3 *There exists a unique linear endomorphism C on the space \mathcal{A} of Bar-Natan diagrams such that:*

$$C(D(\Gamma)) = D(\Gamma) \text{ if } \Gamma \text{ is a primitive graph},\tag{2.11}$$

$$C(D(\Gamma)) = 0 \text{ if } \Gamma \text{ is not prime}.\tag{2.12}$$

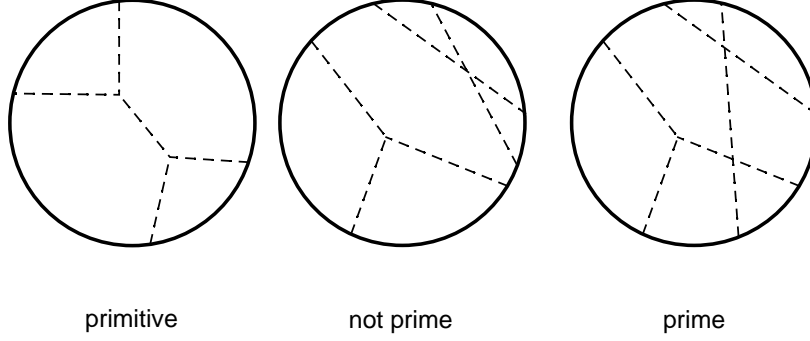


Figure 10:

Proof. This follows from the fact that \mathcal{A} can also be equipped with a cocommutative coproduct, so that it becomes a Hopf algebra, and the primitive graphs correspond to the primitive elements of \mathcal{A} . This implies that \mathcal{A} is isomorphic to the enveloping algebra of the Lie algebra of primitive elements. Thus the conditions of the lemma determine the value of C on the elements $\prod_i D(\Gamma_i)$, Γ_i primitive, spanning \mathcal{A} . \square

3 Feynman rules

Let \mathcal{K} be the space of all smooth embeddings of S^1 into \mathbb{R}^3 . To any vertex-oriented Wilson graph Γ we can associate a differential form on the space \mathcal{K} , denoted by $I(\Gamma)$. The degree of this form is equal to:

$$\text{degree } I(\Gamma) = \sum_{v \in V_\Gamma} (k_v - 3), \quad (3.1)$$

where k_v is the valence of the vertex v . Thus for trivalent graphs, $I(\Gamma)$ is just a function on the space of knots \mathcal{K} . An important property of $I(\Gamma)$ is its behaviour under a change of vertex orientation:

$$I(-\Gamma) = -I(\Gamma). \quad (3.2)$$

The main point of the construction of $I(\Gamma)$ is due to Bott and Taubes. Let us briefly recall their construction. First, to any Wilson graph Γ we can associate a space C_Γ , which is a fiber bundle over the space \mathcal{K} , whose fiber is a compact manifold with corners of dimension $n + 3t$, where n is the number of Wilson vertices of Γ and t is the number of internal vertices:

$$p_\Gamma : C_\Gamma \longrightarrow \mathcal{K}. \quad (3.3)$$

The fiber $p_\Gamma^{-1}(\phi) = \bar{C}_{n,t}(\phi)$, where $\phi : S^1 \rightarrow \mathbb{R}^3 \in \mathcal{K}$ is a knot, is a compactification of the configuration space $C_{n,t}(\phi)$ of n points on the knot and t points in \mathbb{R}^3 :

$$\begin{aligned} C_{n,t}(\phi) &= \{(s_1, \dots, s_n, x_1, \dots, x_t) \in (S^1)^n \times (\mathbb{R}^3)^t \mid \\ &\quad s_1 < \dots < s_n, \text{ cyclically ordered on the circle,} \\ &\quad x_i \neq x_j \text{ if } i \neq j, \\ &\quad \phi(s_i) \neq x_j \text{ if } i \in \{1, \dots, n\}, j \in \{1, \dots, t\}\}. \end{aligned} \quad (3.4)$$

Let us pick two bijections $\beta : V_\Gamma^i \rightarrow \{1, \dots, t\}$, and $\gamma : V_\Gamma^o \rightarrow \{1, \dots, n\}$, such that if $\gamma(u) = i$, and v is the next vertex encountered when going around W_Γ according to its orientation, then $\gamma(v) = i + 1 \bmod n$. Every point of $C_{n,t}(\phi)$ corresponds to an embedding $\Phi : \Gamma \rightarrow \mathbb{R}^3$ (see fig. 11), defined by

$$\Phi(v) = \begin{cases} x_{\beta(v)} & \text{if } v \in V_\Gamma^i, \\ \phi(s_{\gamma(v)}) & \text{if } v \in V_\Gamma^o, \end{cases} \quad (3.5)$$

such that $\Phi(e) = \phi([s_i, s_{i+1}])$ if $e \subset W_\Gamma$, $\partial e = (u, v)$, $\gamma(u) = i$, $\gamma(v) = i + 1$, and if $e \in E_\Gamma^{nW}$, $\Phi(e)$ is the straight line segment joining $\Phi(u)$ to $\Phi(v)$, with $\partial e = (u, v)$. Hence we can identify W_Γ and S^1 via γ , and think of Φ as an extension of ϕ to Γ . Clearly, Φ depends on the choice of β, γ , however $I(\Gamma)$ doesn't.

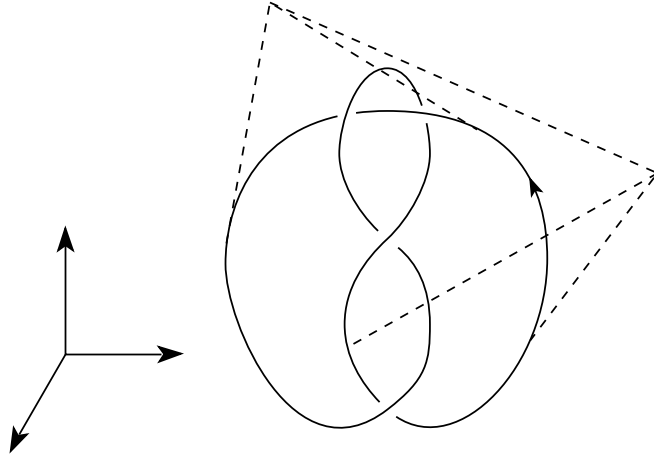


Figure 11:

In (3.8) we omitted the maps β, γ from the notations, thus the indices i, j in x_i, s_j correspond to certain specific vertices in V_Γ^i, V_Γ^o .

Now, if we choose an orientation or_Γ for the edges of Γ , we can define a continuous form on C_Γ of degree $2|E_\Gamma^{nW}|$ ($|E_\Gamma^{nW}|$ is the number of internal edges) as follows:

$$\omega(\Gamma, \text{or}_\Gamma) = \prod_{e \in E_\Gamma^{nW}} \phi_e^* \omega, \quad (3.6)$$

where ω is the Gauss two-form on S^2 :

$$\omega(x) = \frac{1}{4\pi}(x^1 dx^2 \wedge dx^3 + x^3 dx^1 \wedge dx^2 + x^2 dx^3 \wedge dx^1), \quad (3.7)$$

$\phi_e : C_\Gamma \rightarrow S^2$ is given by

$$\begin{cases} u(x_j - x_i) & \text{if } e \text{ is an oriented edge } \partial e = (i, j) \\ & \text{connecting two distinct internal vertices } i, j \in V_\Gamma^i, \\ u(\phi(s_j) - x_i) & \text{if } e \text{ is an oriented edge } \partial e = (i, j) \\ & \text{connecting an internal vertex } i \text{ with an external vertex } j, \\ u(\phi(s_j) - \phi(s_i)) & \text{if } e \text{ is an oriented edge } \partial e = (i, j) \\ & \text{connecting two distinct external vertices } i, j \in V_\Gamma^i, \end{cases} \quad (3.8)$$

and $u : \mathbb{R}^3 - \{0\} \rightarrow S^2$ is defined by

$$u(x) = \frac{x}{|x|} \quad (3.9)$$

for $x = (x^1, x^2, x^3) \in \mathbb{R}^3$.

Remarks:

1. We gave the value of ϕ_e only in the interior of C_Γ . The key point is that the fiber of C_Γ , which is a compact manifold with corners, is the compactification of $C_{n,t}(\phi)$, on which ϕ_e admits a smooth extension.
2. $\omega(\Gamma, -\text{or}_\Gamma) = -\omega(\Gamma, \text{or}_\Gamma) = -\omega(-\Gamma, \text{or}_\Gamma)$.
3. If Γ contains an edge $e \in E_\Gamma^{nW}$ whose boundary consists of only one vertex, $\partial e = (i, i)$, $i \in V_\Gamma$, a case which is not covered by (3.8), it will be convenient to set $\omega(\Gamma, \text{or}_\Gamma) = 0$.

Then if we choose an orientation Ω on the fiber of C_Γ , we can set

$$I(\Gamma, \text{or}_\Gamma, \Omega) = (-1)^{\deg(p_\Gamma)_*\omega} (p_\Gamma)_*\omega(\Gamma, \text{or}_\Gamma), \quad (3.10)$$

where $(p_\Gamma)_*$ is the push-forward (integration along the fiber), whose definition is given in Appendix A. (An orientation is needed in order to integrate forms.) With this definition we see that

$$I(\Gamma, \text{or}_\Gamma, \Omega) = I(-\Gamma, \text{or}_\Gamma, \Omega) = -I(\Gamma, -\text{or}_\Gamma, \Omega) = -I(\Gamma, \text{or}_\Gamma, -\Omega). \quad (3.11)$$

The last two equalities are obvious, the first one follows from the fact that $\omega(\Gamma, \text{or}_\Gamma)$ does not depend on the vertex orientation.

Proposition 2 *If Γ is a Wilson V-oriented graph, which is either trivalent, or trivalent except for one tetravalent vertex, and or_Γ is an orientation of its edges, then there exists an orientation $\Omega(\Gamma, \text{or}_\Gamma)$ on the fiber of C_Γ such that*

$$\Omega(\Gamma, \text{or}_\Gamma) = -\Omega(\Gamma, -\text{or}_\Gamma) = -\Omega(-\Gamma, \text{or}_\Gamma). \quad (3.12)$$

Assuming the proposition, we put

$$I(\Gamma) = I(\Gamma, \text{or}_\Gamma, \Omega(\Gamma, \text{or}_\Gamma)), \quad (3.13)$$

and then (3.2) follows immediately.

It is possible to prove (3.12), and thus to define $I(\Gamma)$, without any assumption on the valences, but we shall refrain from doing that since we don't need to deal with arbitrary graphs in this paper. This would lead to the full definition of the graph complex.

In the case of trivalent graphs, $I(\Gamma)$ is a function on \mathcal{K} , and if $\phi \in \mathcal{K}$

$$I(\Gamma)(\phi) = \int_{C_{n,t}(\phi)} \omega(\Gamma, \text{or}_\Gamma), \quad (3.14)$$

where the orientation on $C_\Gamma(\phi)$ is given by $\Omega(\Gamma, \text{or}_\Gamma)$. The existence of a compactification of $C_\Gamma(\phi)$ on which $\omega(\Gamma, \text{or}_\Gamma)$ extends smoothly tells us that this integral is convergent.

Proof of proposition 2. Let $\phi \in \mathcal{K}$ be a circle embedding, and Γ a trivalent Wilson V-oriented graph equipped with an orientation of the edges or_Γ . The interior $C_{n,t}(\phi)$ of the fiber of C_Γ over ϕ (see (3.4)) is included in $S_1^{V_\Gamma^o} \times (\mathbb{R}^3)^{V_\Gamma^i}$. Here $S_1^{V_\Gamma^o}$ is the set of maps from V_Γ^o to S_1 . Coordinates on this space are denoted by X_v^i , $v \in V_\Gamma$, $i \in \{1, 2, 3\}$. If $v \in V_\Gamma^o$, $i = 1$ and $X_v^1 \in S^1$. If $v \in V_\Gamma^i$, $X_v = (X_v^1, X_v^2, X_v^3) \in \mathbb{R}^3$. Remember the maps o_v , $v \in V_\Gamma$ defining the V-orientation (see 2.4): $o_v = 1$ if $v \in V_\Gamma^o$, $o_v \in \{1, 2, 3\}$ if $v \in V_\Gamma^i$ is trivalent. In the particular case of *trivalent* graphs we define a volume form as follows:

$$\Omega(\Gamma, \text{or}_\Gamma) = \bigwedge_{e \in E_\Gamma^{nW}} \Omega_e, \quad (3.15)$$

$$\Omega_e = dX_v^{o_v(e)} \wedge dX_u^{o_u(e)}, \text{ if } \partial e = (u, v). \quad (3.16)$$

It is straightforward to check that $\Omega(\Gamma, \text{or}_\Gamma)$ does not depend on the choice of maps o_v , $v \in V_\Gamma$, but only on their oriented class.

Now let Γ_x be a V-oriented graph with one tetravalent vertex x . Suppose first that x is an internal vertex, and let e_1, e_2, e_3, e_4 be the four edges surrounding x , with $\partial e_i = (x, v_i)$. The labels are chosen such that $o_x(e_i) = i$. Define $\Omega(\Gamma_x, \text{or}_{\Gamma_x}) = \Omega_1 \wedge \Omega_2$ with

$$\Omega_1 = d^3 X_x \wedge dX_{v_1}^{o_{v_1}(e_1)} \wedge dX_{v_2}^{o_{v_2}(e_2)} \wedge dX_{v_3}^{o_{v_3}(e_3)} \wedge dX_{v_4}^{o_{v_4}(e_4)}, \quad (3.17)$$

where $d^3X = dX^1 \wedge dX^2 \wedge dX^3$, and

$$\Omega_2 = \bigwedge_{\substack{e \in E_{\Gamma_x}^n W \\ \neq e_1, e_2, e_3, e_4}} \Omega_e. \quad (3.18)$$

Next if x is a vertex on the Wilson line, let e_1, e_2 be the two internal edges with $\partial e_i = (x, v_i)$ and w_1, w_2 the two Wilson edges meeting the vertex x . The labels are chosen such that : $o_x(e_i) = i$ and $\partial w_1 = (\cdot, x), \partial w_2 = (x, \cdot)$. Define $\Omega(\Gamma_x, or_{\Gamma_x}) = \Omega_1 \wedge \Omega_2$ with

$$\Omega_1 = dX_x \wedge dX_{v_1}^{o_{v_1}(e_1)} \wedge dX_{v_2}^{o_{v_2}(e_2)}, \quad (3.19)$$

and

$$\Omega_2 = \bigwedge_{\substack{e \in E_{\Gamma_x}^n W \\ \neq e_1, e_2}} \Omega_e. \quad (3.20)$$

By construction $\Omega(\Gamma_x, or_{\Gamma_x})$ does not depend on the choice of maps $o_v, v \in V_{\Gamma}$, but only on their oriented class.

With these definitions, the verification of (3.12) presents no difficulty. \square

4 Invariance under δ

In this section we denote by $|\Gamma|$ and $|\Gamma_+|$ the orders of the groups $\text{Aut}(\Gamma)$ and $\text{Aut}_+(\Gamma)$. Let $G_n^3 = \{\Gamma \in G^3 | \deg \Gamma = n\}$, and

$$Z_n = \sum_{\Gamma \in G_n^3} \frac{1}{|\Gamma|} D(\Gamma) I(\Gamma). \quad (4.1)$$

Thus Z_n is a function on \mathcal{K} with values in \mathcal{A}_n (the space of BN diagrams of order n). We remark that it is independent, as the combination $D(\Gamma)I(\Gamma)$ is, of the choice of vertex orientation for each trivalent Wilson graph. Another observation is that $I(\Gamma)/|\Gamma|$ is a sum over all the embeddings (3.5) of $\Gamma \in G^3$ in \mathbb{R}^3 , weighted by factors $\omega(\Gamma, or_{\Gamma})$.

The expectation value of a Wilson loop is a formal power series in \hbar with values in \mathcal{A} :

$$Z = 1 + \sum_{n=1}^{\infty} \hbar^n Z_n. \quad (4.2)$$

We define

$$\delta I(\Gamma) = \sum_{e \in E_{\Gamma}^a} I(\delta_e \Gamma), \quad (4.3)$$

where the sum is over all admissible edges of Γ . Observe that $\delta I(\Theta) = 0$ according to our Feynman rules, where Θ is the unique Wilson trivalent graph of order 1 represented in fig. 12. Then we have the following theorem, which has been known for some time, see e.g. [12] for a discussion in the setting of the whole graph complex.

Theorem 1

$$\delta Z = 0. \quad (4.4)$$

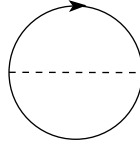


Figure 12:

Proof. We define the following equivalence relation on E_Γ^a :

$$e \sim f \text{ iff } \delta_e \Gamma \sim \pm \delta_f \Gamma \quad (4.5)$$

and denote by $\mathcal{E}_\Gamma = E_\Gamma^a / \sim$ the set of equivalence classes. Let $S_{\Gamma,e}^\pm = \{f \in E_\Gamma^a \mid \delta_f \Gamma \sim \pm \delta_e \Gamma\}$. Observe that if $S_{\Gamma,e}^+ \cap S_{\Gamma,e}^- \neq \emptyset$, then $I(\delta_e \Gamma) = 0$. Then using the definition of δ and the latter property we clearly have:

$$\delta Z_n = \sum_{\Gamma \in G_n^3} \frac{1}{|\Gamma|} D(\Gamma) \sum_{e \in \mathcal{E}_\Gamma} I(\delta_e \Gamma) (|S_{\Gamma,e}^+| - |S_{\Gamma,e}^-|) \quad (4.6)$$

Lemma 4 *If $e \in E_\Gamma^a$ and if e is not as in figs. 13, 14, then*

$$|S_{\Gamma,e}^\pm| = m_\pm(\Gamma, e) \frac{|\Gamma_+|}{|\delta_e \Gamma_+|} \quad (4.7)$$

where $m_\pm(\Gamma, e) = \# i \in \{1, 2, 3\}$ such that $\Gamma_x^{(i)} \sim \pm \Gamma$, with $\Gamma_x = \delta_e \Gamma$.

Assuming this lemma, the proof of theorem 1 goes as follows. First we can assume that $|\Gamma_+| = |\Gamma|$ and $|\delta_e \Gamma_+| = |\delta_e \Gamma|$, otherwise $D(\Gamma) = 0$ or $I(\delta_e \Gamma) = 0$. Let us denote by \mathcal{E}'_Γ the subset of \mathcal{E}_Γ which satisfies the assumption of the lemma. Then

$$\delta Z_n = \sum_{\Gamma \in G_n^3} D(\Gamma) \sum_{e \in \mathcal{E}'_\Gamma} I(\delta_e \Gamma) \frac{1}{|\delta_e \Gamma|} (m_+(\Gamma, e) - m_-(\Gamma, e)), \quad (4.8)$$

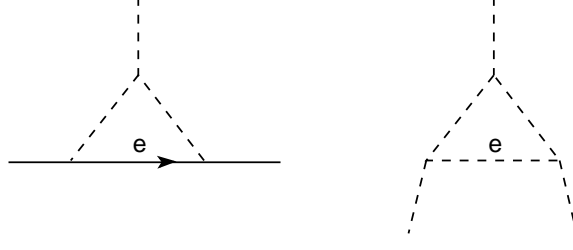


Figure 13:

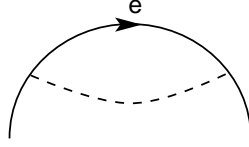


Figure 14:

because $I(\delta_e \Gamma) = 0$ if e is as in figs. 13 or 14. By lemma 1 the coefficient of $I(\Gamma_x)$, where $\Gamma_x = \delta_e \Gamma$, in δZ_n is

$$C(\Gamma_x) = \frac{1}{|\Gamma_x|} \sum D(\Gamma_x^{(i)})((m_+(\Gamma_x^{(i)}, e) - m_-(\Gamma_x^{(i)}, e)), \quad (4.9)$$

where the sum is over the inequivalent $\Gamma_x^{(i)}$ such that $\delta_e \Gamma_x^{(i)} = \Gamma_x$. The relations :

$$\begin{aligned} D(\Gamma_x^{(1)}) + D(\Gamma_x^{(2)}) + D(\Gamma_x^{(3)}) &= 0 \\ D(\Gamma) + D(-\Gamma) &= 0 \end{aligned} \quad (4.10)$$

imply that $C(\Gamma_x) = 0$, which concludes the proof of theorem 1. This can be shown by considering the three cases :

1. All $\Gamma_x^{(i)}$, $i = 1, 2, 3$ are inequivalent.

Then $m_+(\Gamma_x^{(i)}, e) = 1 \forall i$, and $m_-(\Gamma_x^{(i)}, e) = 0 \forall i$. Thus $C(\Gamma_x) = 0$ by the IHX (STU) relations (4.10).

2. $\Gamma^{(1)} \sim \pm \Gamma^{(2)} \neq \Gamma^{(3)}$.

Then $m_+(\Gamma_x^{(3)}, e) = 1$, $m_-(\Gamma_x^{(3)}, e) = 0$

and $m_+(\Gamma_x^{(1)}, e) \in \{1, 2\}$, $m_-(\Gamma_x^{(1)}, e) \in \{0, 1, 2\}$.

We can assume $m_+(\Gamma_x^{(1)}, e) + m_-(\Gamma_x^{(1)}, e) = 2$, otherwise $\Gamma^{(1)} \sim -\Gamma^{(1)}$, so

$D(\Gamma_x^{(1)}) = D(\Gamma_x^{(2)}) = 0$, and $C(\Gamma_x) = D(\Gamma_x^{(3)}) = 0$ by IHX.

If $m_-(\Gamma_x^{(1)}, e) = 0$, $m_+(\Gamma_x^{(1)}, e) = 2$, then $D(\Gamma^{(1)}) = D(\Gamma^{(2)})$ and

$C(\Gamma_x) = 2D(\Gamma^{(1)}) + D(\Gamma^{(3)}) = 0$ by IHX.

If $m_-(\Gamma_x^{(1)}, e) = 1$, $m_+(\Gamma_x^{(1)}, e) = 1$, then $D(\Gamma^{(1)}) + D(\Gamma^{(2)}) = 0$ and $C(\Gamma_x) = D(\Gamma^{(3)}) = 0$ by IHX.

3. $\Gamma^{(1)} \sim \pm \Gamma^{(2)} \sim \pm \Gamma^{(3)}$. Then $D(\Gamma^{(i)}) = 0$, $i = 1, 2, 3$, and $C(\Gamma_x) = 0$ because it is proportional to $D(\Gamma^{(i)})$. \square

Proof of lemma 4. Let

$$\begin{aligned} O_g^\pm(\Gamma, e) &= \{(\rho, \sigma) \in \text{Aut}(V_\Gamma) \times \text{Aut}(E_\Gamma) \mid (\rho, \sigma) \cdot \delta_g \Gamma = \pm \delta_e \Gamma\}, \\ P_i^\pm(\Gamma, e) &= \{(\rho, \sigma) \in \text{Aut}(V_\Gamma) \times \text{Aut}(E_\Gamma) \mid (\rho, \sigma) \cdot \Gamma = \pm \Gamma_x^{(i)}, \Gamma_x = \delta_e \Gamma\}, \end{aligned} \quad (4.11)$$

where $g \in E_\Gamma^a$, $i \in \{1, 2, 3\}$. By proposition 1

$$\left| \bigcup_{g \in E_\Gamma^a} O_g^\pm(\Gamma, e) \right| = \left| \bigcup_{i \in \{1, 2, 3\}} P_i^\pm(\Gamma, e) \right|. \quad (4.12)$$

There is an obvious action of $\text{Aut}_+(\delta_e \Gamma)$ on $O_g^\pm(\Gamma, e)$ which is free and transitive. Therefore, either $O_g^\pm(\Gamma, e) = \emptyset$ or $O_g^\pm(\Gamma, e) \sim \text{Aut}_+(\delta_e \Gamma)$ as a set. By lemma 2, we have

$$g \in S_{\Gamma, e}^\pm \Leftrightarrow O_g^\pm(\Gamma, e) \neq \emptyset, \quad (4.13)$$

so $|S_{\Gamma, e}^\pm| = \#g \in E_\Gamma^a$ s.t. $O_g^\pm(\Gamma, e) \neq \emptyset$, and since $O_g^\pm(\Gamma, e) \cap O_h^\pm(\Gamma, e) = \emptyset$ if $g \neq h$, we get:

$$\left| \bigcup_g O_g^\pm(\Gamma, e) \right| = |S_{\Gamma, e}^\pm| |\delta_e \Gamma_+|. \quad (4.14)$$

Similarly, $\text{Aut}_+(\Gamma)$ acts freely and transitively on $P_i^\pm(\Gamma, e)$ for each $i \in \{1, 2, 3\}$. Moreover we have the following property, whose proof is left to the reader:

Lemma 5 *If e satisfies the assumption of lemma 4, and $i \neq j$,*

$$P_i^\pm(\Gamma, e) \cup P_j^\pm(\Gamma, e) = \emptyset. \quad (4.15)$$

This implies

$$\left| \bigcup_i P_i^\pm(\Gamma, e) \right| = m_\pm(\Gamma, e) |\Gamma_+|. \quad (4.16)$$

The equalities (4.12), (4.14), and (4.16) imply the lemma 4. \square

5 Anomalies

Let Γ be a V-oriented Wilson trivalent graph, and denote by d the exterior differential on \mathcal{K} . Using the definition of $I(\Gamma)$ as a pushforward along the compact fiber of C_Γ , and the

commutation relation between d and the pushforward given in (A.3), the variation of $I(\Gamma)$ under a change of embedding can be expressed as a sum over all the strata of C_Γ [7]:

$$dI(\Gamma) = \sum_{e \in E_\Gamma^a} \epsilon(\delta_e \Gamma) I(\delta_e \Gamma) + \delta_a I(\Gamma), \quad (5.1)$$

where $\epsilon(\delta_e \Gamma) I(\delta_e \Gamma)$ corresponds to the pushforward along the codimension one strata $\partial_e C_\Gamma$, when e is not as in fig. 14, and is obtained by collapsing two vertices of Γ along the admissible edge e . The signs $\epsilon(\delta_e \Gamma) = \pm 1$ depend on the induced orientations of these strata. The “anomalous” term $\delta_a \Gamma$ is the contribution of all the other strata. Bott and Taubes showed that the pushforward along these strata is in general zero except for special ones, which correspond, in the case of prime graphs, to the simultaneous collapse of all the vertices together. Moreover, there exists a “universal” way of calculating the contribution of these strata (universal means here independent of the embedding). More precisely, we can state their result as follows: if Γ is a *prime* V-oriented Wilson trivalent graph, then

$$\delta_a I(\Gamma) = f_\Gamma \frac{dI(\Theta)}{2}. \quad (5.2)$$

Here $dI(\Theta)$ is the differential of the self-linking integral $I(\Theta)$: if ϕ is an embedding of S^1 into \mathbb{R}^3 then

$$I(\Theta)(\phi) = \frac{1}{4\pi} \int_{S^1 \times S^1} ds_1 ds_2 \epsilon_{\mu\nu\rho} \dot{\phi}^\mu(s_1) \dot{\phi}^\nu(s_2) \frac{\phi^\rho(s_2) - \phi^\rho(s_1)}{|\phi(s_2) - \phi(s_1)|^3}. \quad (5.3)$$

The constant of proportionality f_Γ is independent of the embedding ϕ and is expressed as an integral:

$$f_\Gamma = \int_{S_{n,t}} \Theta_\Gamma, \quad (5.4)$$

where $S_{n,t}$ is the following variety of dimension $n + 3t$, n being the number of Wilson vertices and t the number of internal vertices: it is the set of $(a, \eta_1, \dots, \eta_n, \omega_1, \dots, \omega_t) \in S^2 \times \mathbb{R}^n \times (\mathbb{R}^3)^t$ such that:

$$\begin{aligned} & \eta_1 < \dots < \eta_n, \text{ or cyclic permutations} \\ & \omega_i \neq \omega_j \text{ if } i \neq j, \\ & a \cdot \eta_i \neq \omega_j \text{ if } i \in \{1, \dots, n\}, j \in \{1, \dots, t\}, \\ & \sum_{i=1}^n \eta_i^2 + \sum_{i=1}^t |\omega_i|^2 = 1, \\ & \sum_{i=1}^n \eta_i + \sum_{i=1}^t \langle \omega_i, a \rangle = 0. \end{aligned} \quad (5.5)$$

If Γ is equipped with an orientation of its edges and an orientation of its vertices, one defines an orientation $\Omega_{n,t}$ of $S_{n,t}$ induced by $\Omega(\Gamma, or_\Gamma)$, and a $n + 3t$ -form Θ_Γ :

$$\Theta_\Gamma = \prod_{e \in E_\Gamma^{nW}} \Theta_e, \quad (5.6)$$

the product being over all the internal oriented edges of Γ , and $\Theta_e = \phi_e^* \omega$, where ω is the Gauss two-form, and $\phi_e : S_{n,t} \rightarrow S^2$ is given by

$$\begin{cases} u(\omega_j - \omega_i) & \text{if } e \text{ is an oriented edge } (i, j) \text{ connecting two internal vertices } i \text{ and } j, \\ u(a \cdot \eta_j - \omega_i) & \text{if } e \text{ is an oriented edge } (i, j) \text{ connecting an internal vertex } i \\ & \text{with an external vertex } j, \\ u(a(\eta_j - \eta_i)) & \text{if } e \text{ is an oriented edge } (i, j) \text{ connecting two external vertices } i \text{ and } j, \end{cases} \quad (5.7)$$

where u is defined by (3.9).

Note that a simple computation gives $f_\Theta = 2$, which is consistent with our definition of δ and (5.8).

A priori the integral f_Γ is not well-defined because of the singularities at coinciding points. We should verify that the integral is indeed convergent. This follows from the fact that $S_{n,t}$ admits a compactification, which is a manifold with corners, such that Θ_Γ extends smoothly on it.

We would like to emphasize that the definitions of the V-orientation and $\Omega(\Gamma, \text{or}_\Gamma)$ we gave imply that $\epsilon(\delta_e \Gamma) = +1$, hence

Proposition 3 *The total variation of $I(\Gamma)$ is:*

$$dI(\Gamma) = \sum_{e \in E_\Gamma^a} I(\delta_e \Gamma) + \delta_a I(\Gamma). \quad (5.8)$$

Proof. Let Γ be a trivalent graph, $e \in E_\Gamma^a$ an admissible edge, $\bar{C}_{n,t}(\phi)$ the fiber of C_Γ over $\phi \in \mathcal{K}$, and $\partial_e \bar{C}_{n,t}(\phi)$ the codimension one strata of $\bar{C}_{n,t}(\phi)$ corresponding to the collapse of e . Coordinates in $C_{n,t}(\phi)$ can be taken to be

$$((s_i)_{i=1, \dots, n}, (X_i)_{i=1, \dots, t}) \in (S_1)^n \times (\mathbb{R}^3)^t \quad (5.9)$$

If e is an edge on the Wilson line $\partial e = (i+1, i)$, the strata associated to e corresponds to $s_i = s_{i+1}$, and the in-going normal vector field to this stratum is

$$n_e = \frac{\partial}{\partial s_{i+1}} - \frac{\partial}{\partial s_i}, \quad (5.10)$$

since $s_{i+1} > s_i$ in $C_{n,t}(\phi)$. It is clear that

$$i_{n_e} \Omega(\Gamma, \text{or}_\Gamma) = \Omega(\delta_e \Gamma, \text{or}_{\delta_e \Gamma}). \quad (5.11)$$

If e is an internal edge connecting two internal vertices $\partial e = (i, j)$, the vicinity of the stratum associated to e corresponds to $X_j = X + \frac{\alpha}{2}u$, $X_i = X - \frac{\alpha}{2}u$, where $\alpha > 0$ and $u \in S^2$. So the in-going normal vector field to this stratum is $n_e = \frac{\partial}{\partial \alpha}$ and $i_{n_e} \Omega(\Gamma, \text{or}_\Gamma) = \alpha^2 \omega(u) \wedge \Omega(\delta_e \Gamma, \text{or}_{\delta_e \Gamma})$

(ω is the Gauss form). Then using the fact that $\partial_e C_{n,t}(\phi) = S^2 \times C_{n,t-1}(\phi)$ and integrating over S^2 , the induced orientation on $C_{n,t-1}(\phi)$ is $\Omega(\delta_e \Gamma, or_{\delta_e \Gamma})$.

If e is an internal edge connecting one internal vertex and one Wilson vertex a similar analysis can be performed. \square

The vanishing theorems of Bott and Taubes can be improved by showing that:

Theorem 2 (i) *If Γ is not primitive then $f_\Gamma = 0$.*

(ii) *If Γ is primitive and $\deg(\Gamma)$ is even then $f_\Gamma = 0$.*

Proof of (i). Denote by Γ_α the connected subgraphs of $\Gamma - W_\Gamma$,

$$\Gamma - W_\Gamma = \bigcup_{\alpha=1}^n \Gamma_\alpha. \quad (5.12)$$

Consider the following vector fields on $\mathbb{R}^n \times (\mathbb{R}^3)^t$:

$$V_\alpha = \sum_{i \in V_\Gamma^o \cap \Gamma_\alpha} \frac{\partial}{\partial \eta_i} + \sum_{j \in V_\Gamma^i \cap \Gamma_\alpha} \langle a, \frac{\partial}{\partial \omega_j} \rangle, \quad (5.13)$$

$$\mathcal{D} = \sum_{i \in V_\Gamma^o} \eta_i \frac{\partial}{\partial \eta_i} + \sum_{j \in V_\Gamma^i} \langle \omega_j, \frac{\partial}{\partial \omega_j} \rangle. \quad (5.14)$$

V_α is the vector field of translation of all vertices of one subgraph Γ_α , and \mathcal{D} is the vector field of global dilatation. Now put

$$V = \sum_{\alpha} c_\alpha V_\alpha - \lambda \mathcal{D}, \quad (5.15)$$

where c_α are not all simultaneously zero, and satisfy the condition

$$\sum_{\alpha} c_\alpha \deg \Gamma_\alpha = 0, \quad (5.16)$$

and λ is the following function on $S_{n,t}$:

$$\lambda = \sum_{\alpha} c_\alpha \left(\sum_{i \in V_\Gamma^o \cap \Gamma_\alpha} \eta_i + \sum_{j \in V_\Gamma^i \cap \Gamma_\alpha} \langle a, \omega_j \rangle \right). \quad (5.17)$$

Then part (i) of the theorem is a direct consequence of the following two properties:

$$\begin{aligned} \text{If } \Gamma \text{ is not primitive, } V \in TS_{n,t}, V \neq 0 \text{ almost everywhere.} \\ i_V \Theta_\Gamma = 0. \end{aligned} \quad (5.18)$$

Here i_V denotes the interior product. The first property follows from the fact that V preserves the conditions defining the embedding of $S_{n,t}$ into $S^2 \times \mathbb{R}^n \times (\mathbb{R}^3)^t$, the second from $\mathcal{D}(\phi_e) = V_\alpha(\phi_e) = 0$. \square

Proof of (ii). Consider the diffeomorphism S of $S_{n,t}$ defined by:

$$\begin{aligned} S(a) &= -a, \\ S(\eta_i) &= \eta_i, \\ S(\omega_j) &= -\omega_j. \end{aligned} \tag{5.19}$$

We find that the behaviour of Θ_Γ and the orientation of $S_{n,t}$ is given by:

$$\begin{aligned} S^* \Theta_\Gamma &= (-1)^{\frac{n+3t}{2}} \Theta_\Gamma, \\ S^* \Omega_{n,t} &= (-1)^{t+1} \Omega_{n,t}. \end{aligned} \tag{5.20}$$

Thus

$$\int_{S_{n,t}} \Theta_\Gamma = (-1)^{\frac{n+3t}{2}} \int_{S_{n,t}} S^* \Theta_\Gamma = (-1)^{\frac{n+t}{2}+1} \int_{S_{n,t}} \Theta_\Gamma, \tag{5.21}$$

and $f_\Gamma = 0$ if the graph is of even order ($(n+t)/2$ is the order of the graph). \square

So the first non-trivial prime graph which could be anomalous, apart from Θ , appears at order three (this is the only one at this order) and is given in fig. 15.

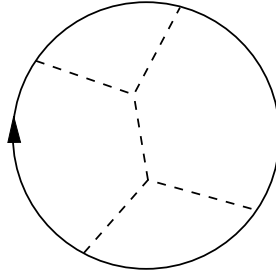


Figure 15:

Combining (5.8) with theorem 1, we can now state the invariance theorem of the expectation value of a Wilson loop. First, let us denote by α the element of \mathcal{A} characterizing the anomalous variations :

$$\alpha = \frac{1}{2} \sum_{\Gamma \text{ primitive}} \hbar^{\deg \Gamma} f_\Gamma \frac{D(\Gamma)}{|\Gamma|}. \tag{5.22}$$

The self-linking integral $I(\Theta)(\phi)$ is not an invariant, but it is well-known [8] that if we introduce a framing, given by a normal vector field ν , then the linking number of the two curves ϕ and $\phi + \nu$ is

$$\text{lk}(\phi, \phi + \nu) = I(\Theta)(\phi) + \tau(\phi, \nu), \tag{5.23}$$

where $\tau(\phi, \nu)$ is the total torsion, given by

$$\tau(\phi, \nu) = \frac{1}{2\pi} \int ds \dot{\phi}(s) \frac{(\dot{\phi}(s), \nu(s), \dot{\nu}(s))}{|\dot{\phi}(s) \wedge \nu(s)|^2}. \quad (5.24)$$

This implies

Theorem 3

$$\hat{Z}(\phi, \nu) = Z(\phi) \exp(\alpha \tau(\phi, \nu)) \quad (5.25)$$

is a framed knot invariant.

In the next section, we are going to show that it is in fact a universal Vassiliev invariant. The theorem is a consequence of the following important property of Z :

Lemma 6 *Let G_*^3 denote the trivalent graphs Γ with $\deg \Gamma > 0$. Then*

$$\log Z = \sum_{\Gamma \in G_*^3} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} C(D(\Gamma)). \quad (5.26)$$

By lemma 3, the map C appearing in this equality is a projector on the subspace of primitive elements of \mathcal{A} . Hence $\log Z$ is a primitive element of the completion $\hat{\mathcal{A}}$ of the Hopf algebra \mathcal{A} . Thus[‡] lemma 6 also shows:

Theorem 4 *$\hat{Z}(\phi, \nu)$ is a group-like element of $\hat{\mathcal{A}}$.*

It is known [6, 14] that the universal Vassiliev invariant constructed from the KZ connection satisfies the same group-like property.

Assuming lemma 6, the proof of theorem 3 is as follows: by theorem 1, part(i) of theorem 2 and (5.2),

$$d \log Z = \frac{dI(\Theta)}{2} \sum_{\Gamma \text{ primitive}} \hbar^{\deg \Gamma} f_{\Gamma} \frac{D(\Gamma)}{|\Gamma|}, \quad (5.27)$$

therefore $d \log \hat{Z} = 0$ by (5.23). \square

At this point, it is perhaps appropriate to mention the behaviour of the invariant under the two operations of reversing the orientation and taking the mirror image of the knot. If

[‡]We owe this remark to T. T. Q. Le.

$\phi \in \mathcal{K}$ is a representative of a knot, denote by $\phi^* = -\phi$ the mirror image of this knot and $\bar{\phi}(s) = \phi(1-s)$ the knot with the opposite orientation. Let $\bar{\Gamma}$ be the graph obtained from Γ by reversing the orientation of the Wilson line. We have the following properties:

$$I(\Gamma)(\phi^*) = (-1)^{\deg \Gamma} I(\Gamma)(\phi), \quad (5.28)$$

$$I(\Gamma)(\bar{\phi}) = (-1)^n I(\bar{\Gamma})(\phi). \quad (5.29)$$

The first property tells us that an invariant of even order appearing in the Chern-Simons expansion cannot distinguish a knot from its mirror image. This property can also be recasted in the usual form:

$$Z(\phi^*)(\hbar) = Z(\phi)(-\hbar). \quad (5.30)$$

From the second property we can deduce that, if the space \mathcal{A} of BN diagrams is such that

$$D(\Gamma) = (-1)^n D(\bar{\Gamma}), \quad (5.31)$$

then the invariant \hat{Z} does not depend on the orientation of knots. It is well-known [5] that this holds in \mathcal{A}^* for all weight systems constructed from simple Lie algebras, but, as far as the authors are aware, whether (5.31) is true or not is still an open question. All these remarks also apply to the universal invariant constructed by Kontsevich.

The remainder of this section is devoted to the proof of lemma 6, which requires some preparation. The main ideas go back to section 9.6 of Bar-Natan's thesis [3]. First, we need to define *marked graphs*. They are pairs (Γ, e) , where Γ is a trivalent graph and $e \in E_\Gamma^W$ is an edge of W_Γ . The edge e is also called the marking of (Γ, e) . Marked graphs are depicted as in fig. 16. Recall that for trivalent graphs Γ , $I(\Gamma)$ is an integral over the configuration

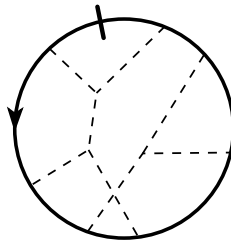


Figure 16:

space $C_{n,t}$, where points on the circle are cyclically ordered. For marked graphs (Γ, e) , we define $I(\Gamma, e)$ to be the same integral, but taken over a configuration space $L_{n,t}$, where we consider the points on the circle linearly ordered ($0 < s_1 < \dots < s_n < 1$) and the marked Wilson edge corresponds to the interval $[s_n, s_1]$. Thus, by definition we have:

$$I(\Gamma) = \sum_{e \in E_\Gamma^W} I(\Gamma, e). \quad (5.32)$$

The group $\text{Aut}(\Gamma)$ acts on E_Γ^W , and if $g \in \text{Aut}(\Gamma)$, $I(\Gamma, g \cdot e) = I(\Gamma, e)$. Let $\text{Aut}(\Gamma)_e$ be the stabilizer of $e \in E_\Gamma^W$, and $\Gamma' = \Gamma - W_\Gamma$ be the interior of Γ . It is easy to check that

$$\text{Aut}(\Gamma)_e = \text{Aut}(\Gamma)_{e'} = \text{Aut}(\Gamma'), \quad (5.33)$$

for all $e, e' \in E_\Gamma^W$. Let $M(\Gamma) = E_\Gamma^W / \text{Aut}(\Gamma)$ be the set of equivalence classes of markings of Γ . The number of markings in the class of any $e \in E_\Gamma^W$ is $|\text{Aut}(\Gamma) / \text{Aut}(\Gamma')| = |\Gamma| / |\Gamma'|$. Therefore,

$$\frac{I(\Gamma)}{|\Gamma|} = \sum_{e \in M(\Gamma)} \frac{I(\Gamma, e)}{|\Gamma'|}. \quad (5.34)$$

Given two marked graphs (Γ_1, e_1) and (Γ_2, e_2) , with n_1 and n_2 external vertices, and a (n_1, n_2) -shuffle σ , we can construct a new marked graph $(\Gamma, e) = (\Gamma_1, e_1) \times_\sigma (\Gamma_2, e_2)$, which we call a shuffle product. A (n_1, n_2) -shuffle is a permutation σ of $\{1, \dots, n_1 + n_2\}$ such that $\sigma(1) < \dots < \sigma(n_1)$ and $\sigma(n_1 + 1) < \dots < \sigma(n_1 + n_2)$. The set of (n_1, n_2) -shuffles will be denoted by $\Sigma(n_1, n_2)$. The shuffle product is defined as follows: label the external vertices of (Γ_1, e_1) by $1, 2, \dots, n_1$, going around the Wilson line in the sense given by its orientation, so that for the oriented edge e_1 , $\partial e_1 = (n_1, 1)$, for the next edge e'_1 , $\partial e'_1 = (1, 2)$, etc. Similarly, label the external vertices of (Γ_2, e_2) by $n_1 + 1, \dots, n_1 + n_2$, with $\partial e_2 = (n_1 + n_2, n_1 + 1)$. The interior of the shuffle product is $\Gamma' = \Gamma'_1 \cup \Gamma'_2$. Going around the Wilson line, its external vertices are $\sigma^{-1}(1), \sigma^{-1}(2), \dots, \sigma^{-1}(n_1 + n_2)$, and the marked edge is $\partial e = (\sigma^{-1}(n_1 + n_2), \sigma^{-1}(1))$. This is illustrated in fig. 17. Now it should be fairly obvious

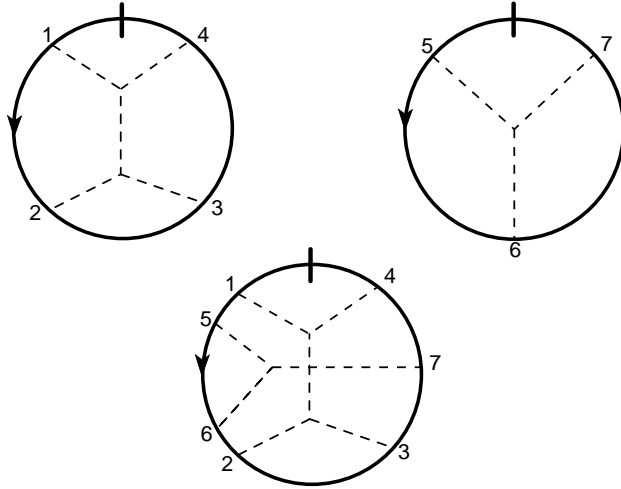


Figure 17: $\sigma(1, 2, 3, 4, 5, 6, 7) = (1, 4, 5, 7, 2, 3, 6)$

that

$$I(\Gamma_1, e_1) I(\Gamma_2, e_2) = \sum_{\sigma \in \Sigma(n_1, n_2)} I((\Gamma_1, e_1) \times_\sigma (\Gamma_2, e_2)). \quad (5.35)$$

Let $S(\Gamma_1, \Gamma_2) = M(\Gamma_1) \times M(\Gamma_2) \times \Sigma(n_1, n_2)$. We have a map $s : S(\Gamma_1, \Gamma_2) \ni (e_1, e_2, \sigma) \mapsto (\Gamma_1, e_1) \times_\sigma (\Gamma_2, e_2)$. Let

$$P(\Gamma|\Gamma_1, \Gamma_2) = \{t \in S(\Gamma_1, \Gamma_2) \mid s(t) = (\Gamma, e), \text{ for some marking } e\}, \quad (5.36)$$

and $n(\Gamma|\Gamma_1, \Gamma_2) = |P(\Gamma|\Gamma_1, \Gamma_2)|$. Elements of $P(\Gamma|\Gamma_1, \Gamma_2)$ will be called partitions of Γ in two parts Γ_1, Γ_2 . Clearly, partitions can be defined for an arbitrary number of parts.

Lemma 7 *Let $\Gamma_1, \dots, \Gamma_m \in G^3$, then*

$$\frac{I(\Gamma_1)}{|\Gamma_1|} \dots \frac{I(\Gamma_m)}{|\Gamma_m|} = \sum_{\Gamma \in G^3} \frac{I(\Gamma)}{|\Gamma|} n(\Gamma|\Gamma_1, \dots, \Gamma_m). \quad (5.37)$$

Proof. We can restrict ourselves to the case $m = 2$. Using (5.33), (5.34) and (5.35) we get

$$\frac{I(\Gamma_1)}{|\Gamma_1|} \frac{I(\Gamma_2)}{|\Gamma_2|} = \frac{1}{|\Gamma'_1||\Gamma'_2|} \sum_{e_1 \in M(\Gamma_1)} \sum_{e_2 \in M(\Gamma_2)} \sum_{\sigma \in \Sigma(n_1, n_2)} I((\Gamma_1, e_1) \times_\sigma (\Gamma_2, e_2)). \quad (5.38)$$

Observe that if $(\Gamma, e) = (\Gamma_1, e_1) \times_\sigma (\Gamma_2, e_2)$, $\text{Aut}(\Gamma') = \text{Aut}(\Gamma'_1) \times \text{Aut}(\Gamma'_2)$, so that

$$|\Gamma'_1||\Gamma'_2| = |\Gamma'|. \quad (5.39)$$

Consider the map $\tau : S(\Gamma_1, \Gamma_2) \rightarrow S(\Gamma_1, \Gamma_2)$ defined by fig. 18. It is invertible and generates

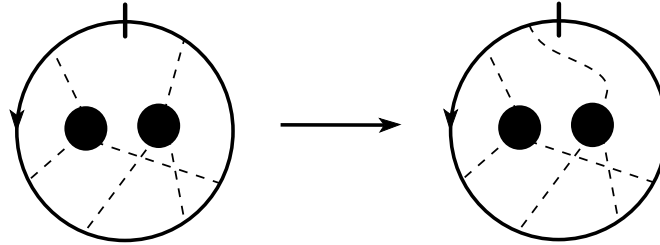


Figure 18:

a finite cyclic group G of order at most $n_1 + n_2$. There is also a transformation, which we denote by the same letter τ , acting directly on marked graphs: $\tau \cdot (\Gamma, e) = (\Gamma, e')$, where e' is the marking directly adjacent to e . Moreover, $\tau \circ s = s \circ \tau$.

The sum on the r.h.s. of (5.38) can be rearranged into a sum over the orbits of G in $S(\Gamma_1, \Gamma_2)$. If $(e_1, e_2, \sigma) \in S(\Gamma_1, \Gamma_2)$, $s((e_1, e_2, \sigma)) = (\Gamma, e)$, then the orbit $G \cdot (e_1, e_2, \sigma)$ covers a certain number of times, say $n(e_1, e_2, \sigma)$, the orbit $G \cdot (\Gamma, e)$. It follows that the contribution of one orbit is:

$$\sum_{(f_1, f_2, \mu) \in G \cdot (e_1, e_2, \sigma)} I((\Gamma_1, f_1) \times_\mu (\Gamma_2, f_2)) = n(e_1, e_2, \sigma) \sum_{e \in M(\Gamma)} I(\Gamma, e). \quad (5.40)$$

Finally, if we add the contributions of all the orbits in $S(\Gamma_1, \Gamma_2)$ which cover the same $G \cdot (\Gamma, e)$, we get

$$n(\Gamma|\Gamma_1, \Gamma_2) \sum_{e \in M(\Gamma)} I(\Gamma, e), \quad (5.41)$$

where $n(\Gamma|\Gamma_1, \Gamma_2)$ is the sum of all the factors $n(e_1, e_2, \sigma)$. Taking (5.39) into account and using (5.34) once again, the proof is completed. \square

Proof of lemma 6. Writing

$$Z = 1 + \sum_{\Gamma \in G_*^3} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} D(\Gamma), \quad (5.42)$$

and using the product in \mathcal{A} , we define $\log Z$ by the formal power series expansion

$$\log Z = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{\Gamma \in G_*^3} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} D(\Gamma) \right)^m. \quad (5.43)$$

Now from lemma 7 we get

$$\begin{aligned} \left(\sum_{\Gamma \in G_*^3} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} D(\Gamma) \right)^m &= \sum_{\Gamma_1, \dots, \Gamma_m} \frac{I(\Gamma_1)}{|\Gamma_1|} \dots \frac{I(\Gamma_m)}{|\Gamma_m|} \prod_{j=1}^m D(\Gamma_j) \hbar^{\deg \Gamma_j} \\ &= \sum_{\Gamma} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} \sum_{\Gamma_1, \dots, \Gamma_m} n(\Gamma|\Gamma_1, \dots, \Gamma_m) \prod_{j=1}^m D(\Gamma_j). \end{aligned} \quad (5.44)$$

Thus, $\log Z$ becomes

$$\log Z = \sum_{\Gamma \in G_*^3} \hbar^{\deg \Gamma} \frac{I(\Gamma)}{|\Gamma|} c(\Gamma), \quad (5.45)$$

where

$$c(\Gamma) = \sum_{\text{partitions } P} \frac{(-1)^{|P|+1}}{|P|} D(P), \quad (5.46)$$

the sum is over all partitions P of Γ , $|P|$ is the number of parts, and $D(P) = \prod_{j=1}^m D(\Gamma_j)$ if P is a partition with $|P| = m$ parts $\Gamma_1, \dots, \Gamma_m$. It remains to show that $c(\Gamma) = C(D(\Gamma))$. It is clear that if Γ is primitive, $c(\Gamma) = D(\Gamma)$. If we can prove that $c(\Gamma) = 0$ when Γ is not prime, then by lemma 3 we are done. Bar-Natan has shown in [3] how to achieve this in three steps, which we reproduce here since his thesis has not been published.

Step 1. We have $n(\Gamma|\Gamma_1, \Gamma_2) = n(\Gamma|\Gamma_2, \Gamma_1)$, and more generally $n(\Gamma|\Gamma_1, \Gamma_2, \dots, \Gamma_m) = n(\Gamma|\Gamma_m, \Gamma_1, \dots, \Gamma_{m-1})$. Define a cyclic partition of Γ to be a partition modulo a cyclic permutation of the parts. Then the total number of partitions into m parts is divisible by m and is the number of cyclic partitions into m parts. Thus we can rewrite (5.46) as

$$c(\Gamma) = \sum_{\text{cyclic partitions } P} (-1)^{|P|+1} D(P). \quad (5.47)$$

Step 2. We show that $c(\Gamma) = 0$ if Γ is the connected sum along the Wilson lines of two graphs Λ and M , such that Λ is primitive. To do this, we construct an involution ρ on the set of cyclic partitions of Γ such that $|\rho(P)| = |P| \pm 1$ and $D(\rho(P)) = D(P)$. Take a cyclic partition $P = \{\Gamma_1, \dots, \Gamma_m\}$. We can always assume that $\Lambda \subset \Gamma_1$. If $\Lambda = \Gamma_1$, we put $\rho(P) = \{\Lambda \cup \Gamma_2, \dots, \Gamma_m\}$, otherwise Λ is properly contained in Γ_1 and we put $\rho(P) = \{\Lambda, \Gamma_1 - \Lambda, \Gamma_2, \dots, \Gamma_m\}$.

Step 3. We show that $c(S) = c(T) - c(U)$, where S, T, U are the three graphs appearing in the STU relation of [5]. The argument is the following: to every partition of S there corresponds a partition of T and U , but the converse is not true. However, the only partitions of T and U to which one cannot associate a partition of S are those where the two edges not contained in the Wilson line belong to two distinct parts. Call these the exceptional partitions. There is an obvious 1-1 correspondence between the exceptional partitions of T and U , which preserves the number of parts. Hence the contributions of the exceptional partitions cancel in $c(T) - c(U)$.

Now by step 3, there is a map $\tilde{c} : \mathcal{A} \rightarrow \mathcal{A}$ such that $c(\Gamma) = \tilde{c}(D(\Gamma))$, for all $\Gamma \in G^3$. If Γ is not prime, $D(\Gamma) = D(\Lambda)D(M)$, with both factors non-trivial. But in \mathcal{A} , $D(\Gamma)$ is a linear combination of products of primitive elements, therefore $c(\Gamma) = 0$ by step 2. \square .

6 \hat{Z} is a universal Vassiliev invariant

In this section we show that \hat{Z} is a universal Vassiliev Invariant. First of all, let us briefly recall what this means. Put $\hat{Z}(K) = \hat{Z}(\phi, \nu)$, where K is a framed knot, and

$$\hat{Z}(K) = \sum_{N \geq 0} \hat{Z}_N(K) \hbar^N. \quad (6.1)$$

We extend $\hat{Z}(K)$ to singular knots in the usual way: if K^j is a singular knot with j double points, we define $\hat{Z}(K^j)$ by

$$\hat{Z}_N(K^j) = \hat{Z}_N(K_+^{j-1}) - \hat{Z}_N(K_-^{j-1}), \quad (6.2)$$

where K_{\pm}^{j-1} are two knots with $j - 1$ double points obtained by desingularizing one of the double points of K^j as shown in fig. 19. We also need the fact that each singular knot K^j defines a unique chord diagram $\Gamma(K^j)$ of degree j [5]. A formal power series $\Phi(K) = \sum_{N \geq 0} \Phi_N(K) \hbar^N \in \mathcal{A}[[\hbar]]$ is a universal Vassiliev invariant if it satisfies the following two properties:

U1 $\Phi_N(K^j) = 0$ if $j > N$, i.e. $\Phi_N(K)$ is a Vassiliev invariant of degree $\leq N$,

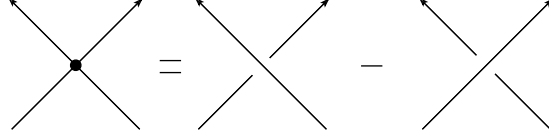


Figure 19:

U2 $\Phi_N(K^N) = D(\Gamma(K^N))$, i.e. $\Phi(K^N) - \hbar^N D(\Gamma(K^N))$ is divisible by \hbar^{N+1} .

Theorem 5 $\hat{Z}(K)$ is a universal Vassiliev invariant.

Proof. If we desingularize the double points of K^j in all possible ways, we obtain 2^j knots $K_{\vec{\varepsilon}}$, where $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_j)$ and $\varepsilon_i = \pm 1$ for all $1 \leq i \leq j$. It is possible to choose the corresponding embeddings and framings $\phi_{\vec{\varepsilon}}$ and $\nu_{\vec{\varepsilon}}$ such that the total torsion $\tau(\phi_{\vec{\varepsilon}}, \nu_{\vec{\varepsilon}})$ is independent of $\vec{\varepsilon}$. Thus in the forthcoming arguments, we can ignore the factor containing the torsion in $\hat{Z}(K)$.

For non-singular K , $\hat{Z}_N(K)$ is a sum of contributions $I(K, \Gamma)$ indexed by graphs Γ of degree $N = (n + t)/2$, where n is the number of (external) vertices on the Wilson line W_Γ and t is the number of (internal) trivalent vertices.

By (6.2), we have for each Γ a contribution $I(K^j, \Gamma)$ to $\hat{Z}_N(K^j)$, which is an alternating sum of 2^j terms $I(K_{\vec{\varepsilon}}, \Gamma)$, corresponding to all possible desingularizations of K^j . Let $\vec{\varepsilon}(i)$ denote the vector obtained from $\vec{\varepsilon}$ by changing the sign of the i -th coordinate. Then for $1 \leq k \leq j$,

$$\begin{aligned} I(K^j, \Gamma) &= \sum_{\vec{\varepsilon}} \left(\prod_{i=1}^j \varepsilon_i \right) I(K_{\vec{\varepsilon}}, \Gamma) \\ &= \sum_{\vec{\varepsilon}, \varepsilon_k=1} \left(\prod_{i \neq k} \varepsilon_i \right) (I(K_{\vec{\varepsilon}}, \Gamma) - I(K_{\vec{\varepsilon}(k)}, \Gamma)). \end{aligned} \quad (6.3)$$

Now $I(K_{\vec{\varepsilon}}, \Gamma)$ is the integral of a function of n real variables s_α , $1 \leq \alpha \leq n$, which are parameters along the knot $K_{\vec{\varepsilon}}$, and t points of \mathbb{R}^3 . Let R_i , $1 \leq i \leq j$ be a small ball containing the i -th singular point of K^j . Let $C_{\vec{\varepsilon}, i} = R_i \cap K_{\vec{\varepsilon}}$. By definition,

$$K_{\vec{\varepsilon}} - C_{\vec{\varepsilon}, i} = K_{\vec{\varepsilon}(i)} - C_{\vec{\varepsilon}(i), i}, \quad (6.4)$$

for each i . Let

$$D_i = \{s \in [0, 1] \mid K^j(s) \in R_i\}. \quad (6.5)$$

Remembering the discussion surrounding (3.5), we can identify D_i with a subset of W_Γ . Let us say that D_i is *occupied* if it contains at least one of the n external vertices. Let $U_{\vec{\varepsilon}}$ be

the integration domain of $I(K_{\bar{\varepsilon}}, \Gamma)$, $E_k = \{s_\alpha \notin D_k, \forall \alpha\}$. If $F_k \subset E_k$, then $U_{\bar{\varepsilon}} \cap F_k$ is a sub-domain with D_k not occupied. Denote by $I(K_{\bar{\varepsilon}}, \Gamma, F_k)$ the contribution of $U_{\bar{\varepsilon}} \cap F_k$ to $I(K_{\bar{\varepsilon}}, \Gamma)$. We can also define $I_k(K^j, \Gamma, F_k)$, replacing $I(\cdot, \Gamma)$ by $I(\cdot, \Gamma, F_k)$ everywhere in the first equality in (6.3).

Lemma 8 *If $j \geq k \geq 1$ and $F_k \subset E_k$, $I(K^j, \Gamma, F_k) = 0$.*

Proof. By (6.4) and the second equality of (6.3), the lemma becomes obvious. \square

This lemma already implies that $\hat{Z}_N(K)$ is a finite type invariant. Indeed, it says that all non-zero contributions to $I(K^j, \Gamma)$ come from the domain \mathcal{D}_Γ where all the j regions D_i are occupied. But if Γ is such that $j > n$, then $\mathcal{D}_\Gamma = \emptyset$ and $I(K^j, \Gamma) = 0$. Since for all Γ of degree N , $2N \geq n$, we get $\hat{Z}_N(K^j) = 0$ if $j > 2N$. We need to work a little bit more to prove the theorem.

Put $\bar{D} = W_\Gamma - \bigcup_i D_i$. It corresponds to the common part of all the $K_{\bar{\varepsilon}}$. Let $\Gamma' = \Gamma - \{\text{edges of } W_\Gamma\}$ be the interior of Γ .

Lemma 9 *If Γ is a graph of degree N with $j > N$, and every D_i is occupied, then there exist D_i and D_m , $i \neq m$, such that there is a path in Γ' going from D_i to D_m .*

Proof. Let Γ_i be the connected component of Γ' containing all vertices from D_i . If there is no pair (i, m) , $i \neq m$ with a path from D_i to D_m , then for all i , the external vertices of Γ_i all belong to D_i or \bar{D} , but never to D_m with $m \neq i$. Thus the graphs Γ_i are all disjoint. Since every Γ_i has at least two vertices, and there are j such graphs, $N \geq (2j)/2 = j$. \square

For a while, let us forget knots and consider embeddings in \mathbb{R}^3 of connected graphs Γ , whose vertices are either trivalent (internal) or univalent (external). Assign fixed locations $x_\alpha \in \mathbb{R}^3$, $\alpha = 1, \dots, n$ to the external vertices of Γ . Integrating over all the internal vertices the form $\omega(\Gamma, \text{or } \Gamma)$, we obtain a real-valued n -form $g_\Gamma(x)$, where $x = (x_1, \dots, x_n)$.

Lemma 10 *Let $x_{\alpha\beta} = x_\alpha - x_\beta$. If $\alpha \neq \beta$,*

$$\lim_{|x_{\alpha\beta}| \rightarrow \infty} g_\Gamma(x) = 0.$$

Proof. Consider a path in Γ connecting the two vertices x_α and x_β . If this path consists of a single edge, the lemma is trivial, so we assume that it passes through internal vertices z_1, z_2, \dots, z_k , with $k \geq 1$, which are numbered in such a way that z_j is the vertex reached after traveling through j consecutive edges, starting from x_α .

Now make the substitution $z_j \rightarrow z_j + x_\beta$ of the integration variables. Then in g_Γ the z_1 integral is $g_Y(x_{\alpha\beta}, z_2, w)$, where w is the end of the third edge connected to z_1 , and g_Y is the form corresponding to the graph Y . Fortunately, g_Y was computed explicitly in [10], and looking at the expression one can see easily that $g_Y(x_{\alpha\beta}, z_2, w) \rightarrow 0$ as $|x_{\alpha\beta}| \rightarrow \infty$. \square

U1 now follows from the last two lemmas: by the invariance of $\hat{Z}_N(K)$, we can assume that the balls R_i are all very far from each other. Then by lemma 9, each term $I(K_{\vec{\varepsilon}}, \Gamma)$ contributing to $\hat{Z}_N(K^j)$ will contain a factor $g_\Delta(x)$, where Δ is a connected subgraph of Γ' having two external vertices which are very far apart, and by lemma 10 it vanishes.

Let us now prove U2. We use the same notations as in the proof of lemma 9. If there is a path in Γ' connecting disjoint regions $D_i, D_m, m \neq i$, then by lemma 10 $I(K^j, \Gamma) = 0$. Thus the only graphs Γ which contribute to $\hat{Z}_N(K^j)$ are those such that all the subgraphs Γ_i are disjoint. Since $N = j$, every Γ_i has exactly two vertices.

Using the invariance of $\hat{Z}_N(K)$, we can assume that for each i , there is a very large neighborhood V_i of R_i in which the two connected components of $L_{\vec{\varepsilon}, i} = K_{\vec{\varepsilon}} \cap V_i$ are both planar curves, for all $\vec{\varepsilon}$. Then if for some value of i , the unique edge of Γ_i joins two points of the same component of $L_{\vec{\varepsilon}, i}$, $I(K^j, \Gamma) = 0$, by planarity. Therefore all Γ_i join two distinct connected components of $L_{\vec{\varepsilon}, i}$. Thus we see that the only non-vanishing contribution to $\hat{Z}_N(K^j)$ for $N = j$ comes from the graph $\Gamma(K^j)$. It is easy to see that each V_i will contribute a factor 1 to a product decomposition of $I(K^j, \Gamma(K^j))$ corresponding to the j pairs of integration variables s_α . This concludes the proof of the theorem. \square

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Appendix A Integration along the fiber

Let (M, B, p) be a bundle $(p : M \rightarrow B)$ such that the fiber $p^{-1}(x) = F_x, x \in B$, is a compact finite-dimensional oriented (possibly stratified) manifold. The push-forward, or integration along the fiber, is a linear morphism p_*^M from the space of $(n + p)$ -differential forms on M , to the space of p -differential forms on B , where n is the dimension of the fiber. It is defined as follows: let $x \in B$ and X_1, \dots, X_p be p vectors in $T_x B$, let ω be a $(n + p)$ -form on M , then the push-forward is

$$(p_*^M \omega)_x(X_1, \dots, X_p) = \int_{F_x} i_{\tilde{X}_p} \cdots i_{\tilde{X}_1} \omega, \quad (\text{A.1})$$

where i denotes the interior product, and \tilde{X}_i is any lift of X_i as a section of TM over F_x . This definition is independent of the lift's choice. This integration along the fiber is a direct generalization of usual integration of forms on compact manifolds: if ω is a n -form then the push-forward is

$$(p_*^M \omega) = \int_{F_x} \omega, \quad (\text{A.2})$$

and B appears in this case as a parameter space. An important property of the push-forward is its commutation with the differential operator, leading to the generalization of Stokes theorem:

$$d_B p_*^M \omega = p_*^M d_M \omega + (-1)^{\deg(p_*^M \omega)} p_*^{\partial M} \omega. \quad (\text{A.3})$$

Here d_B (resp. d_M) denotes the differential operator on B (resp. M). In order to define the integration $p_*^{\partial M}$, we need to provide the boundary of the fiber ∂F_x with an orientation (in the case of a stratified space the boundary is the codimension 1 strata). This orientation is induced by the orientation on the whole fiber as follows: let Ω be an orientation (i.e. an n -form) on the fiber F_x , and n_e the in-going normal vector field of the boundary. The boundary orientation is defined by $i_{n_e} \Omega$. The formula (A.3) holds with this convention. Let us remark that it is not the usual orientation convention of the boundary, for if we apply (A.3) to an n -form ω on a trivial bundle $M = B \times F$ which does not depend on the parameter space B , then the l.h.s. of (A.3) is zero, and this equation reduces to Stokes' formula

$$\int_F d\omega = - \int_{\partial F} \omega. \quad (\text{A.4})$$

Another important property of the push-forward is its behaviour with respect to the pullback. If ϕ is a form on B , ω a form on M , and $p^*(\phi)$ denotes the pullback of ϕ on M , then

$$p_*^M (p^*(\phi) \wedge \omega) = \phi \wedge p_*^M (\omega). \quad (\text{A.5})$$

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